Mean-Variance Analysis: Risk vs. Expected Return

Introduction

The goal of this analysis is explaining how an agent might determine the optimal portfolio given a variety of available assets. The particular model considered here determines the optimal portfolio for an investor allocating money between two assets, one that is risk-free and one that is risky. The analysis for more than one risky asset is more complex, but similar in spirit.

From the results, we can explain how agents' optimal portfolios will change when the risks and expected returns of assets change. These changes cause reallocations of investment funds, which explains why agents do not simply adopt optimal portfolios and hold them indefinitely. Instead, they trade as new information becomes available.

The Model

This analysis will be conducted on a diagram of risk (measured in terms of the standard deviation of returns) and expected return. The diagram is a (mirror image) version of the Two Goods - Two Prices diagram, with expected return on the vertical axis and risk on the horizontal axis. It is a mirror image because agents do not place a positive value on risk, but rather value safety, which is the complement to risk. Safety is not a good choice for the horizontal axis because, while the empirical counterpart to zero risk is often taken to be short-term government securities, the notion of zero safety has no obvious empirical interpretation.

For a single risky asset, the agent has a "budget constraint" that is a straight line through the two assets as they are located on the diagram of risk vs. expected return. Here, the risk-free asset costs 100 and returns 105 next year. The risky asset costs 100 and has an expected return next year of 110 with a standard deviation of 50.

This analysis assumes that there is a know tradeoff for a given investor between risk (as measured by the variance of returns) and expected return. The "Utility-Based Valuation of Risk" application derives this tradeoff for a specific utility function.

Here, the agent's utility function is

$$U = W^a/a$$

where W is dollars and U is utility. The parameter a determines his risk aversion and, hence, the indifference curves for his preferences with respect to risk and expected return. If a = 0.5, we have the square root utility function considered in "Utility-Based Valuation of Risk."

Exercises

1. Locate four points on the agent's budget constraint. Draw the budget constraint.

2. Draw three indifference curves that pass through risk-free returns of 105, 110, and 115.

3. Find the highest possible indifference curve reachable from the agent's budget constraint. Draw a dot at the location of the optimal portfolio.

4. For the utility function parameter a equal to 0.1, 0.5, and 0.8, find the optimal portfolio. Does risk aversion increase or decrease with increases in a?

5. For the original case of a = 0.5, find the optimal portfolio with the original risk-free return RF of 105. Increase the risk-free return RF to 107. Local the new optimal portfolio. Why did the agent move toward the risk-free asset?

6. Graph the effect on the optimal portfolio of changes in the expected return EX of the risky asset.

7. Graph the effect on the optimal portfolio of changes in the risk SD of the risky asset.